Theory of subglacial hydrology

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Structure of the lecture

- Glacier hydrology in general
- Subglacial drainage:
  physical processes
  mathematical description
  numerical models: Schoof's & GlaDS
- Outlook: coupling to ice flow and erosion

Asides:

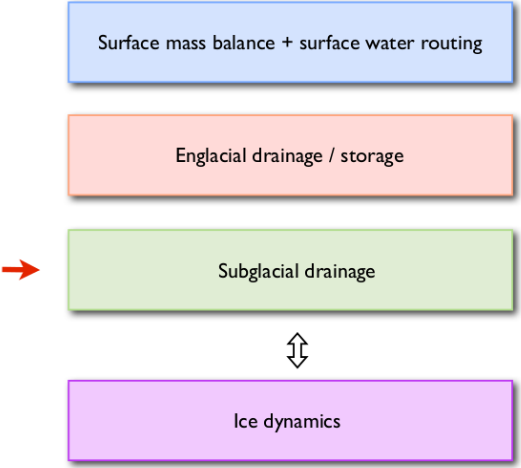
- many slides by Ian Hewitt (the nice looking ones)
- who has Matlab installed?

Motivation to model glacier drainage

Why bother?

- provides input for the basal boundary conditions for ice flow models
- meltwater contributes to sea water convection under ice shelves and in fjords
- hazard assessment of glacier lake outburst floods
- subglacial erosion and sediment transport
- transport of tracers/nutrients/microbes

Glacier hydrology



- Temperature index models
- Energy balance models
- Refreezing / routing

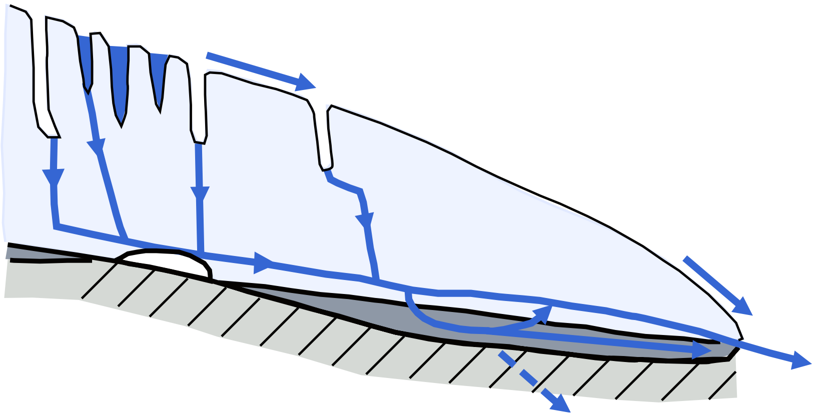
- Crevasses
- Moulins
- Storage / refreezing

- Basal melting / refreezing
- Storage
- Transport

- Sliding law
- Thermodynamics

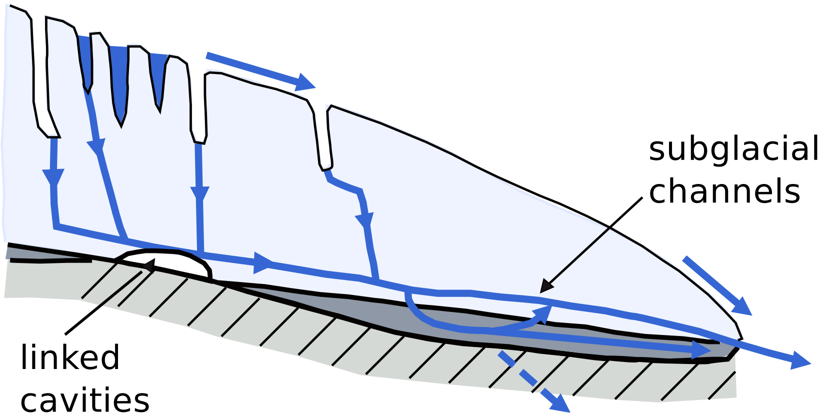
Water flow through a glacier

Cross-section of ablation area of glacier/ice sheet



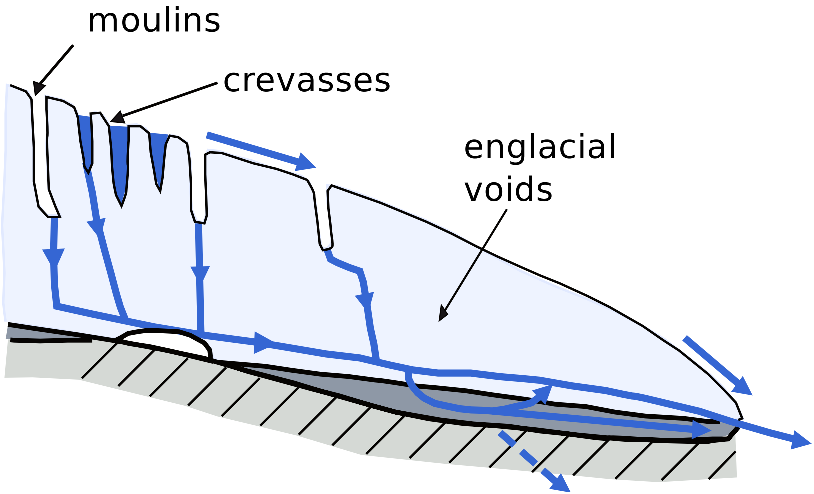
Water flow through a glacier

Subglacial drainage system



Water flow through a glacier

Englacial drainage system



Key principles to model water flow

For the simplest kind of hydrology modelling three ingredients are needed:

1) Conservation of water mass

2) Water flows down the hydraulic potential

$$\underbrace{\phi}_{\text{hydraulic potential}} = \underbrace{p_w}_{\text{pressure potential}} + \underbrace{\rho_w g H}_{\text{elevation potential}}$$

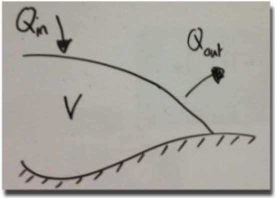
with the discharge

$$q \propto -\sqrt{\nabla \phi}$$

3) Time evolution of drainage space, e.g. channel x-sectional area

This ignores many processes, e.g.: water temperature, Navier-Stokes flow

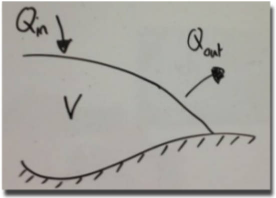
Key principles:
(1) conservation of water mass



Rate of change of volume = flux in - flux out

$$\frac{dV}{dt} = Q_{in} - Q_{out}$$

Key principles:
(1) conservation of water mass



Rate of change of volume = flux in - flux out

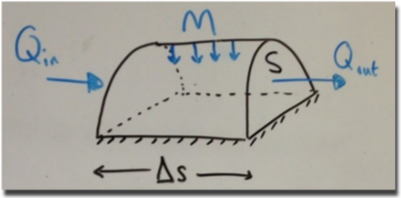
$$\frac{dV}{dt} = Q_{in} - Q_{out}$$

Rate of change of volume = flux in - flux out + source

$$\frac{dV}{dt} = Q_{in} - Q_{out} + M\Delta s$$

$$V = S\Delta s \quad \Rightarrow \quad \frac{\partial S}{\partial t} + \frac{Q_{out} - Q_{in}}{\Delta s} = M$$

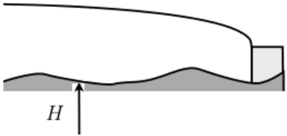
$$\Delta s \to 0 \quad \Rightarrow \quad \boxed{\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = M}$$



Key principles: (2) water flow

Hydraulic potential:

$$\underbrace{\phi}_{\text{hydraulic potential}} = \underbrace{p_w}_{\text{pressure potential}} + \underbrace{\rho_w g H}_{\text{elevation potential}}$$



Manning or Darcy-Weisbach formula relates ϕ to the water flow:

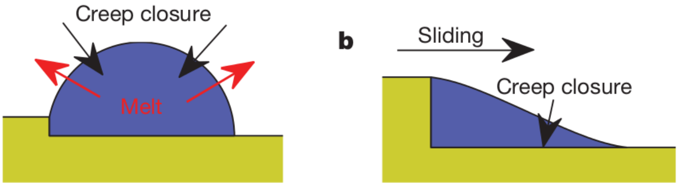
$$q \propto -\sqrt{\nabla \phi}$$

(for turbulent flow)

Effective pressure: ice overburden pressure - water pressure

$$N = p_i - p_w$$

Key principles:
(3) time evolution of drainage space



(Schoof 2010)

This is where the physical characteristics of a particular type of drainage system feature.

I will look at these as I discuss different drainage types.

Drainage types

Subglacial drainage can occur through both a distributed as well as through a channelised system.

Channelised

- R-channels
- canals
- Nye-channels

Distributed

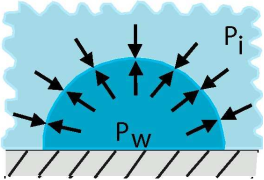
- sheet flow
- linked cavities
- micro cavities
- through till

Lakes

Channels

R-channels are incised into the ice

(Röthlisberger 1972, Shreve 1972, Nye 1976, Spring & Hutter 1982)



1) mass conservation $$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = m_C$$

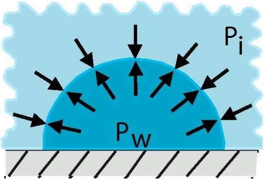
2) turbulent flow $$Q = -k_c S^\alpha \frac{\partial \phi}{\partial s} \left| \frac{\partial \phi}{\partial s} \right|^{-1/2}$$

3) opening and closure $$\frac{\partial S}{\partial t} = \frac{\Xi - \Pi}{\rho_i L} - v_{cC}(\phi, S)$$

Unknowns:
Q channel discharge
ϕ hydraulic potential
S channel x-sectional area

Channel opening and closure

3) opening and closure $\frac{\partial S}{\partial t} = \frac{\Xi - \Pi}{\rho_i L} - v_{cC}(\phi, S)$



Closure is due to ice creep

$$v_{cC}(N,S) = \tilde{A}S|N|^{n-1}N$$

Opening is due to dissipation of potential energy in the water flow:

$$\Xi = -Q\frac{\partial \phi}{\partial s}$$

Pressure melting point effects can lead to both opening and closure:

$$\Pi = -c_t c_w \rho_w 0.3 \qquad Q\frac{\partial p_w}{\partial s}(\phi - \phi_m)$$

Channel opening and closure

3) opening and closure $$\frac{\partial S}{\partial t} = \frac{\Xi - \Pi}{\rho_i L} - v_{cC}(\phi, S)$$

Closure is due to ice creep

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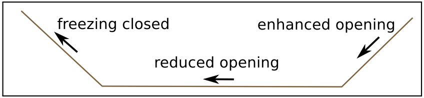

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Pressure melting point effects can lead to both opening and closure:

$$\Pi = -c_t c_w \rho_w 0.3 \qquad Q \frac{\partial p_w}{\partial s} (\phi - \phi_m)$$

R-Channel characteristics

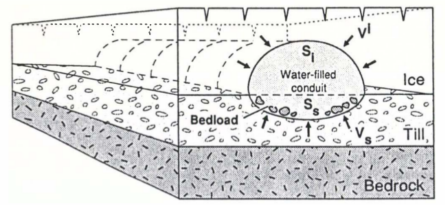
In steady state: effective pressure increases with discharge.

This means that larger channels capture the discharge of smaller channels:
Thus they form an arborescent network.

Canals

If there is a sediment bed then channels can incise both into the ice and into the sediment

(Walder & Fowler 1994, Ng 1998)



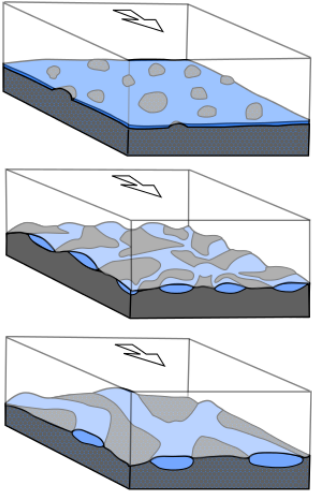
When they are mostly incised into the sediment, then effective pressure decreases with discharge.

When they are mostly incised into the ice, then (as for R-channels) effective pressure increases with discharge.

Distributed drainage

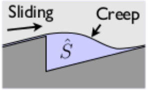
'Distributed' systems

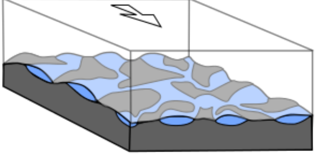
- uneven water films
  Weertman 1972, Walder 1982, Alley 1989, Creyts & Schoof 2009
- micro-cavity networks
  Fountain & Walder 1998, Flowers & Clarke 2002
- canals
  Walder & Fowler 1994
- linked cavities
  Lliboutry 1976, Walder 1986, Fowler 1986, Kamb 1987



Linked cavities

Lliboutry 1968, Walder 1986, Kamb 1987, Schoof 2010





Cavities open due to sliding

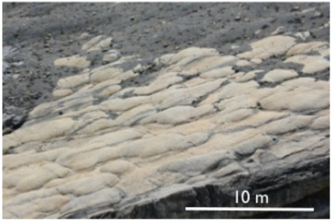
& close due to viscous creep

$$\frac{\partial \hat{S}}{\partial t} = U_b h_r - \hat{A}\hat{S}N^n$$

Bump height — Includes shape factor

Local flow rate $\hat{Q} = -\hat{K}\hat{S}^{\alpha}\hat{\mathcal{G}}^{1/2}$

Local potential gradient

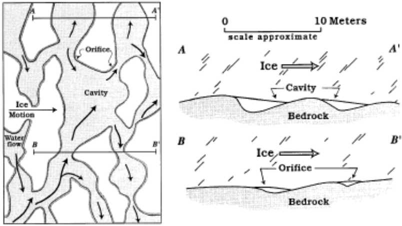


Smaller 'orifices' control the flow

Wall melting can be significant in the orifices

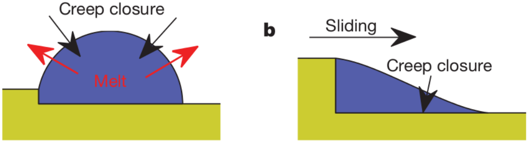
Can lead to 'unstable' growth

⇨ This is how a channel starts to form



Fountain & Walder 1998, Kamb 1987

The cavity formulation can be combined with the R-channel equations (Kessler & Anderson 2004, Schoof 2010)



(Schoof 2010)

Same as R-channels plus one extra term:

1) mass conservation $$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = m_C$$

2) turbulent flow $$Q = -k_c S^{\alpha} \frac{\partial \phi}{\partial s} \left| \frac{\partial \phi}{\partial s} \right|^{-1/2}$$

3) opening and closure $$\frac{\partial S}{\partial t} = \frac{\Xi - \Pi}{\rho_i L} - v_{cC}(\phi, S) + u_b\, h_r$$

Sheet flow

Distributed flow is *distributed*: model it in 2D!

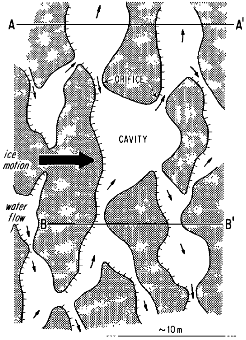
Porous sheet consisting of linked cavities
$S \to h$:

1) mass conservation $$\frac{\partial h}{\partial t} + \nabla.\mathbf{q} = m$$

2) turbulent flow $$\mathbf{q} = -kh^{\alpha}\nabla\phi\,|\nabla\phi|^{-1/2}$$

3) opening and closure $$\frac{\partial h}{\partial t} = v_o(u_b, h) - v_c(\phi, h)$$

Unknowns:
q sheet discharge
ϕ hydraulic potential
h sheet thickness



Kamb 1987

Sheet opening and closure

3) opening and closure $$\frac{\partial h}{\partial t} = v_o(u_b, h) - v_c(\phi, h)$$

Hewitt (2011)

Opening is due to the ice sliding over the bumpy bed

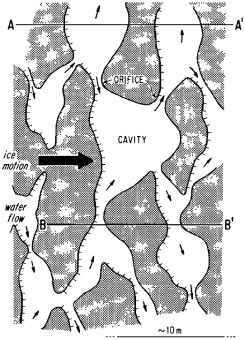
$$v_o(u_b, h) = \frac{u_b}{l_r}(h_r - h)$$

with speed u_b.

Closure is due to ice creep

$$v_c(N, h) = \tilde{A} h |N|^{n-1} N$$

with effective pressure $N = \phi_0 - \phi$.



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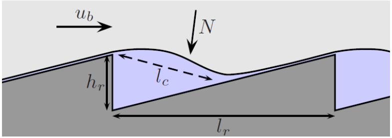

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$$v_c(N, h) = \tilde{A} h |N|^{n-1} N$$

with effective pressure $N = \phi_0 - \phi$.

Combining channelised and distributed flow

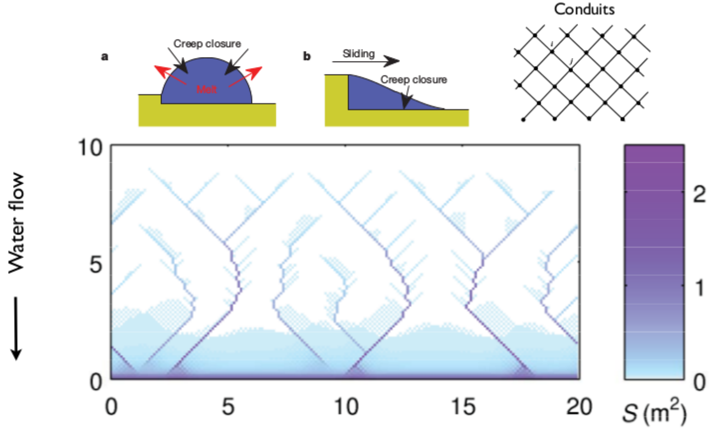
We want to model the subglacial drainage system on the whole bed, i.e. in **two dimensions**.

Channels (or at least their equations) are 1D beasts.
Sheet flow can be formulated in 2D

How to combined them?

Network of conduits

Combine conduits into a network (Schoof 2010)



Schoof 2010

Combining R-channels with sheet model: GlaDS

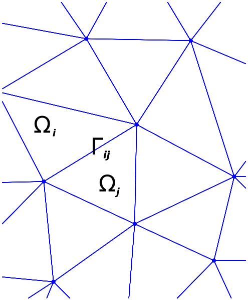
The 1D R-channel equations can be combined with the 2D sheet equations (Hewitt & al 2012, Hewitt 2013, Werder & al 2013)

	Sheet (2D)	**R-channels (1D)**
1) Mass conserv.	$\frac{\partial h}{\partial t} + \nabla.\mathbf{q} = m$	$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{\Xi - \Pi}{\rho_w L} + m_C$
2) Turbulent flow	$\mathbf{q} = -k_s h^\alpha \left\lvert \nabla \phi \right\rvert^{-1/2} \nabla \phi$	$Q = -k_C S^\alpha \left\lvert \frac{\partial \phi}{\partial s} \right\rvert^{-1/2} \frac{\partial \phi}{\partial s}$
3) Time evolution	$\frac{\partial h}{\partial t} = v_o - v_c$	$\frac{\partial S}{\partial t} = \frac{\Xi - \Pi}{\rho_i L} - v_{cC}$

Coupled **2D** model

A network of *potential* R-channels is put on top of the sheet:

- channels on network edges Γ_{ij}
- sheet in-between channels Ω_i
- water conservation
  - at network nodes
  - in the exchange between channel and sheet

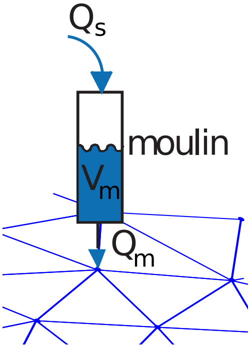


Englacial storage and transport

Water storage:

water stored $\propto$ *water pressure*

Moulins:



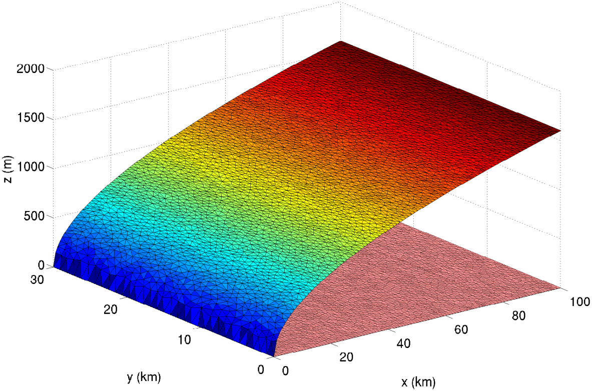
Example results

Showing results of running GlaDS on:

- synthetic ice sheet margin
- synthetic valley glacier

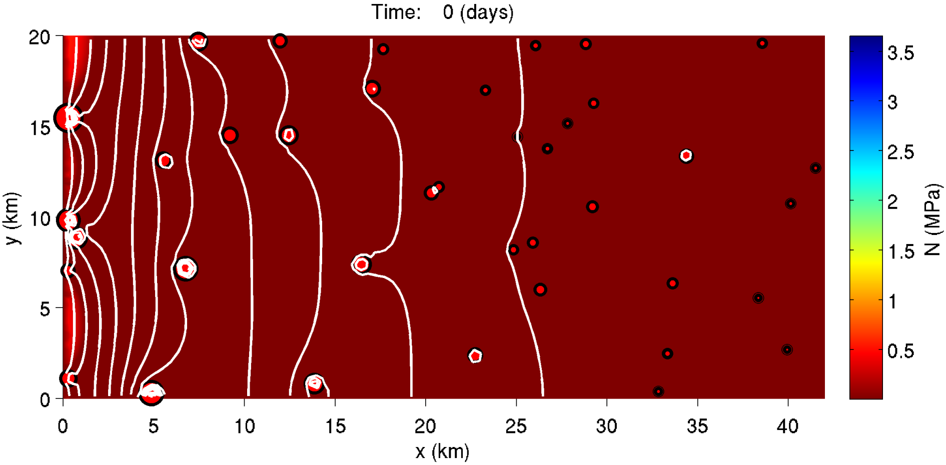
Results: synthetic topography

20 km × 60 km, square root surface glacier:



Model run from Werder et al. (JGR 2013)

Run to steady state: channels (lines), effective pressure (color) and hydraulic potential (contours)



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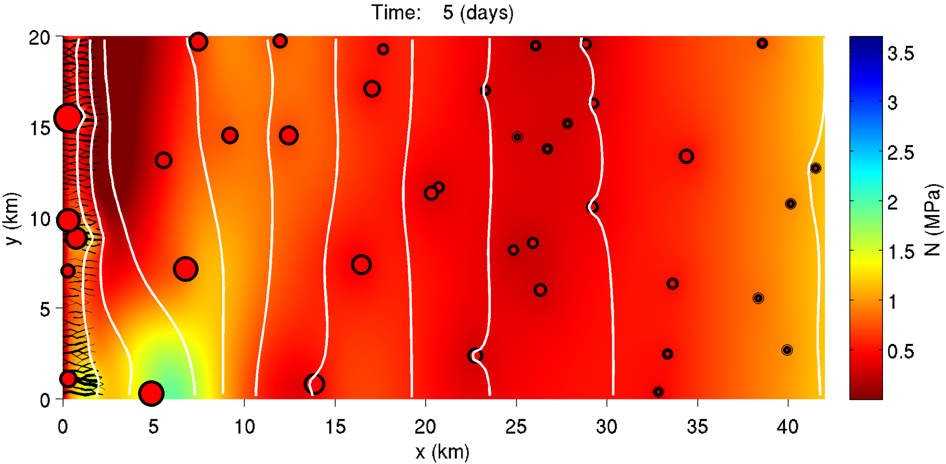

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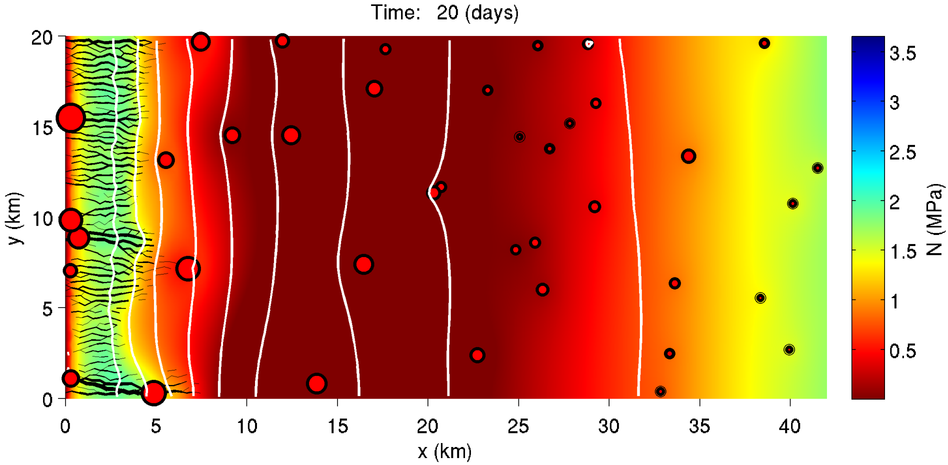

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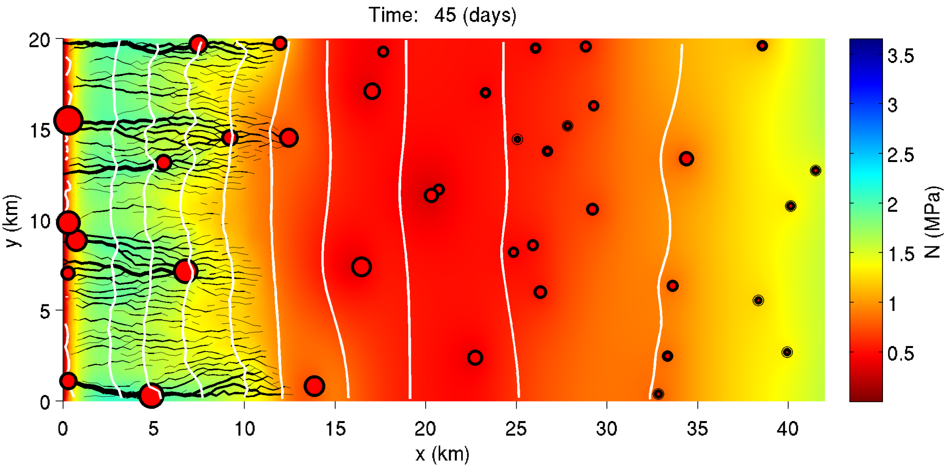

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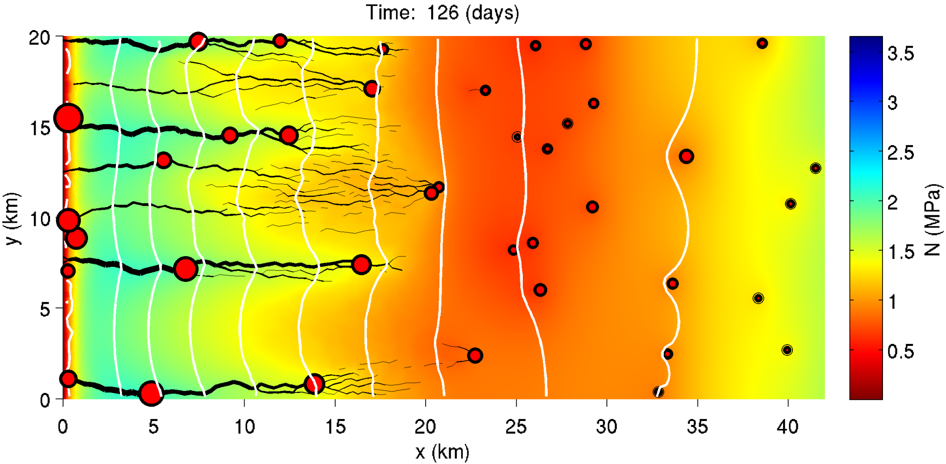

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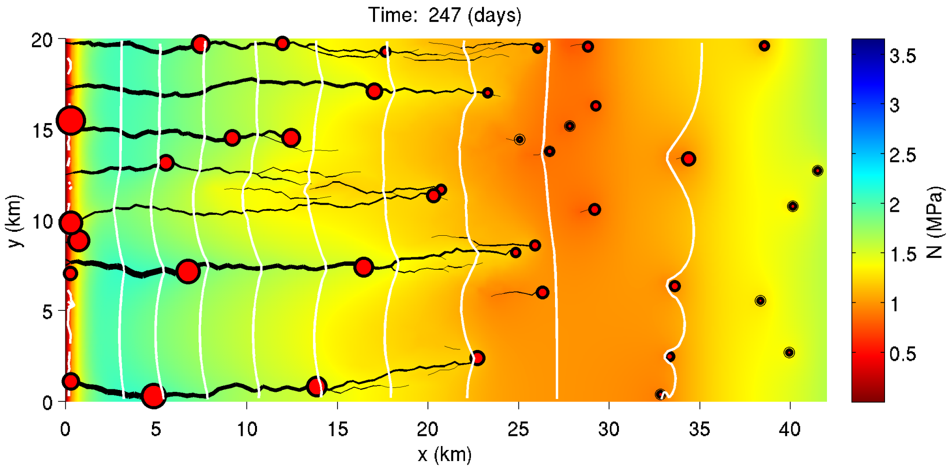

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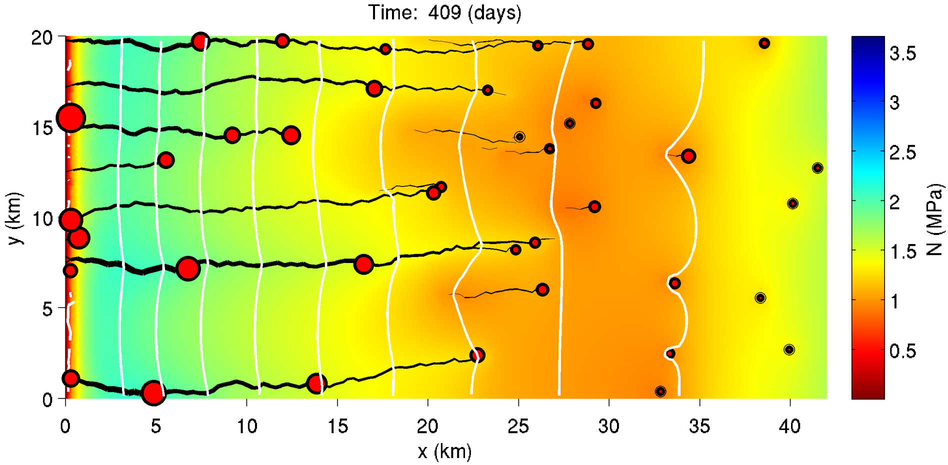

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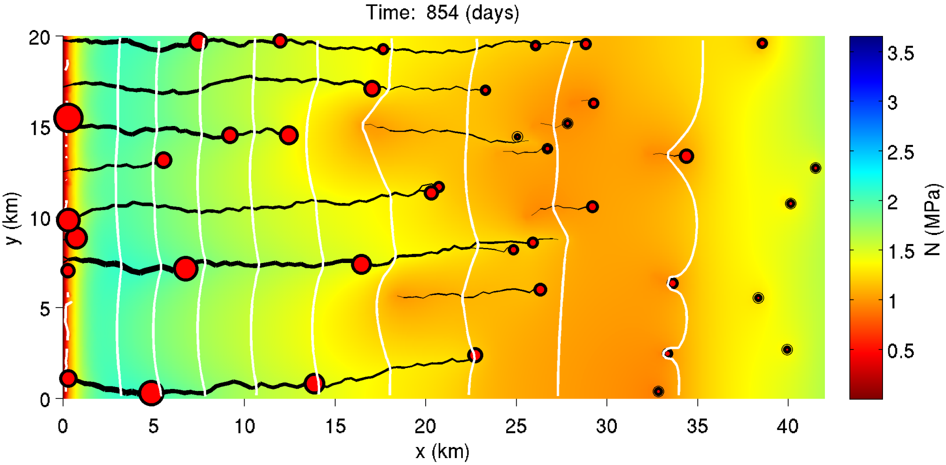

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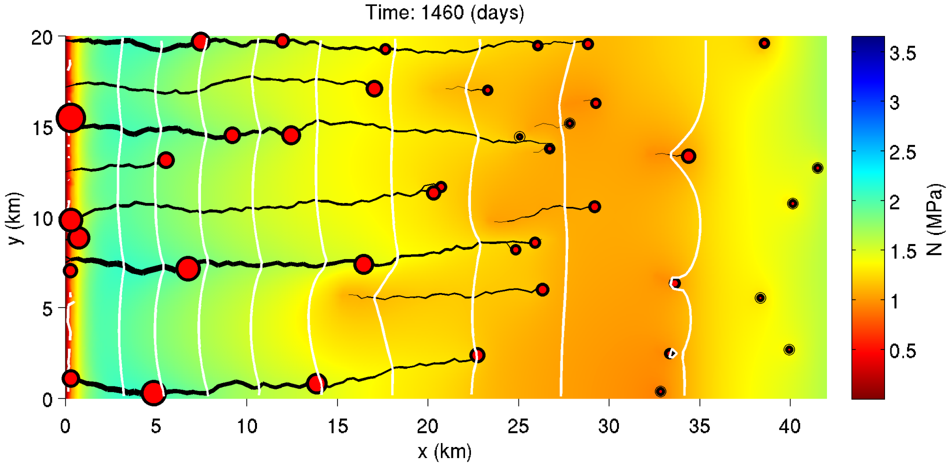

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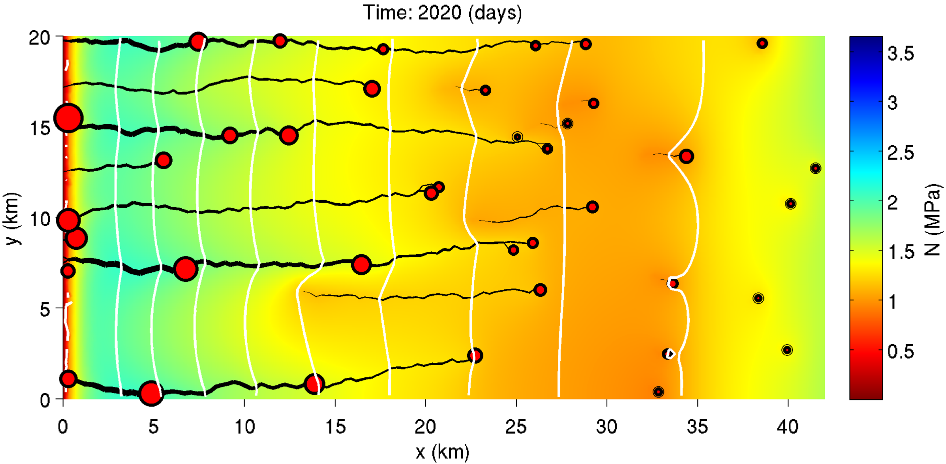

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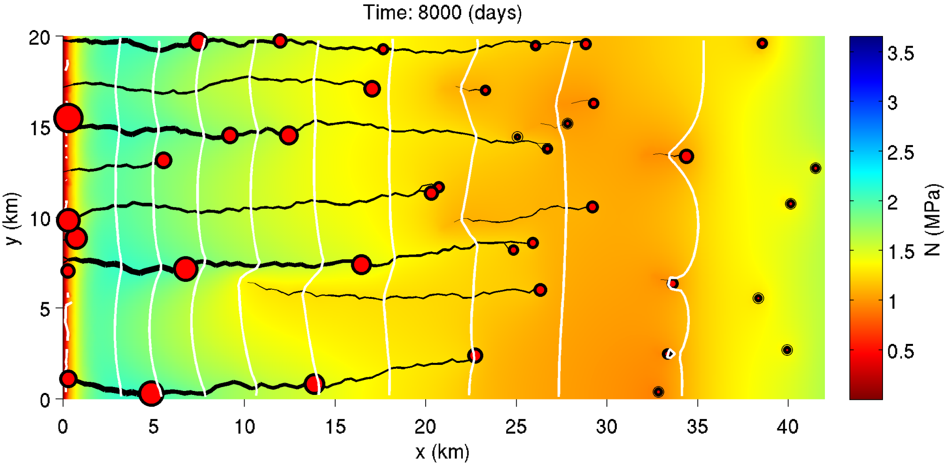
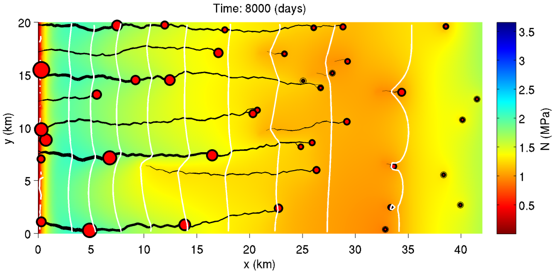


Synthetic topography summary



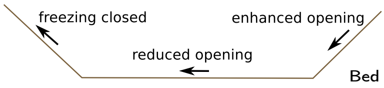
- distributed flow where discharge is low
- channel network links moulins
- channel network is formed as part of the model solution
- generally lower pressure along channels
- some channels peter out which have high pressure

Pressure-melt term effects: revisited

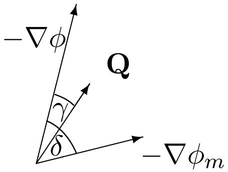
Melting point of water is dependent on pressure.

3) channel opening and closure $\dfrac{\partial S}{\partial t} = \dfrac{\Xi - \Pi}{\rho_i L} - v_{cC}(\phi, S)$

Π depends on bed slope $\nabla\phi_m$: $\Pi = -0.3\,\mathbf{Q}\,.\,\nabla(\phi - \phi_m)$



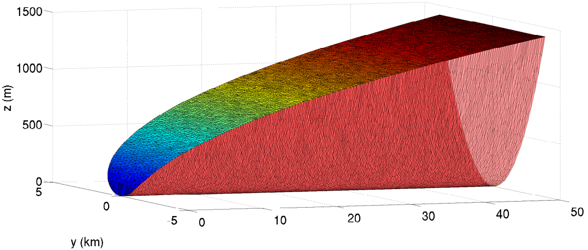
Channel shutdown on steep adverse slopes



Wonky channels

Pressure-melt effects

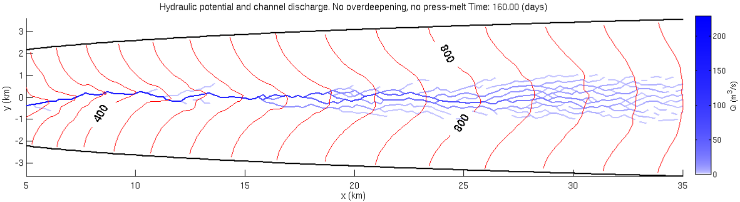
Model application to a synthetic valley glacier 10 kmx50 km



Forced with uniform input into the sheet of 5cm/day $\approx$ 230 m^3/s total

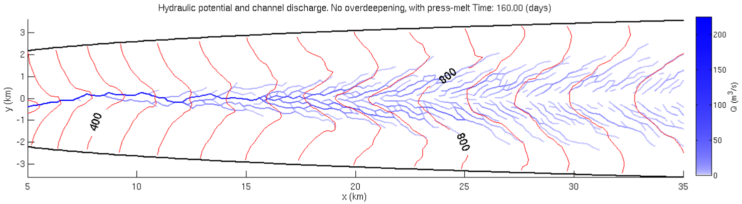
Run to steady state

Animation of channel and hydraulic potential:
no pressure-melt term

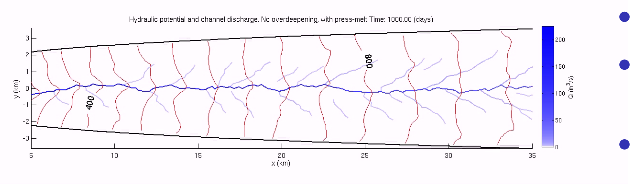


Run to steady state

Animation of channel and hydraulic potential:
with pressure-melt term

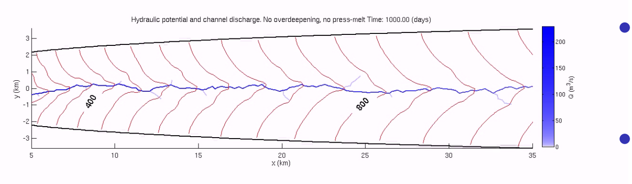


Pressure melt term summary



With pressure melt term:

- increases average pressure
- equalises pressure across trough
- **wonky** side channels at an angle to hyd. potential

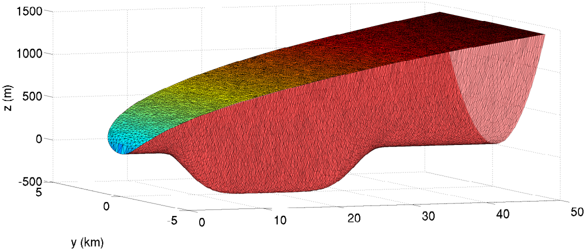


No pressure melt term:

- main channel incises a pronounced valley into the hydraulic potential
- channels perpendicular to hyd. potential

An overdeepened glacier

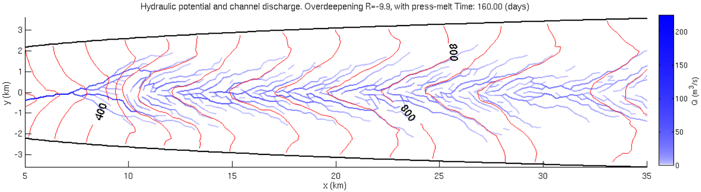
Model application to same synthetic glacier but with overdeepened bed:



20 km long overdeepening
Same forcing

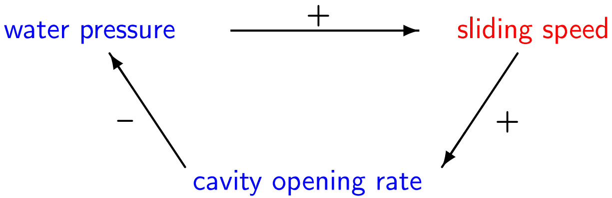
R=-9.9

Animation of channel and hydraulic potential:
with pressure-melt term, with overdeepening



Coupling to ice flow

Two way coupling between subglacial hydrology and ice flow:



The canonical example are the *spring events*.

Coupling to ice flow

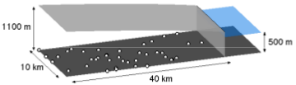
Ian Hewitt has hydrology coupled to ice dynamics in his model

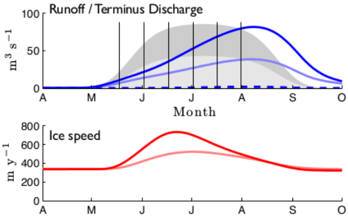
(Hewitt 2013)

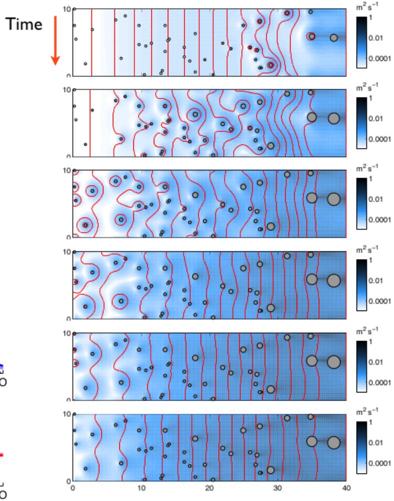
Conduit + cavity drainage

Sliding law $\tau_b = C U_b^{1/3} N^{1/3}$

Ice flow due to sliding only (SSA)







Coupling to erosion

Subglacial erosion and sediment transport are strongly dependent on hydrology
$\rightarrow$ add erosion to GlaDS

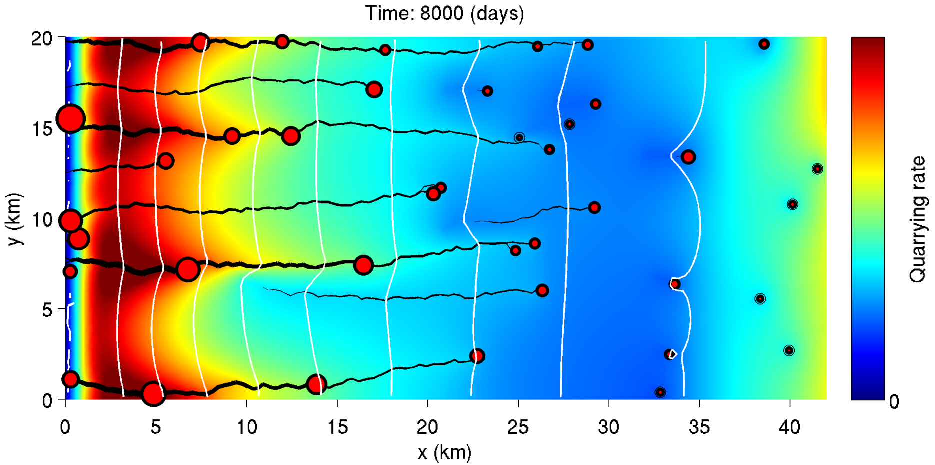
Erosion rates according to **quarrying law of Iverson** (2012):

$$\dot{E}_b \propto u_b \, h \, k(h, N)$$

u_b sliding speed
h sheet thickness
N effective pressure N
k quarrying probability

- h and N are calculated by GlaDS
- u_b is taken as constant
  (i.e. no ice flow model coupled yet)

Coupling to erosion: quarrying



Finished!

To finish you off, here the combined sheet-channels equations again:

	Sheet (2D)	**R-channels (1D)**
1) Mass conserv.	$\frac{\partial h}{\partial t} + \nabla.\mathbf{q} = m$	$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{\Xi - \Pi}{\rho_w L} + m_C$
2) Turbulent flow	$\mathbf{q} = -k_s h^\alpha \left\vert\nabla\phi\right\vert^{-1/2} \nabla\phi$	$Q = -k_C S^\alpha \left\vert\frac{\partial \phi}{\partial s}\right\vert^{-1/2} \frac{\partial \phi}{\partial s}$
3) Time evolution	$\frac{\partial h}{\partial t} = v_o - v_c$	$\frac{\partial S}{\partial t} = \frac{\Xi - \Pi}{\rho_i L} - v_{cC}$

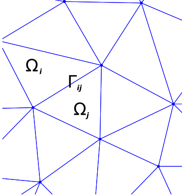
If anyone is interested in working with GlaDS or Ian's model, just ask!

For exercises, download code from `http://tinyurl.com/gl-1Dhydro` and unzip.

Summary of equations

Equations of linked cavity sheet and R-channels (no storage):

	Sheet	**R-channels**
Mass conserv.	$\frac{\partial h}{\partial t} + \nabla.\mathbf{q} = m$	$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{\Xi - \Pi}{\rho_w L} + m_C$
Turbulent flow	$\mathbf{q} = -k_s h^\alpha \left\vert\nabla\phi\right\vert^{\beta-2} \nabla\phi$	$Q = -k_C S^\alpha \left\vert\frac{\partial \phi}{\partial s}\right\vert^{\beta-2} \frac{\partial \phi}{\partial s}$
Time evolution	$\frac{\partial h}{\partial t} = v_o - v_c$	$\frac{\partial S}{\partial t} = \frac{\Xi - \Pi}{\rho_i L} - v_{cC}$
Opening	$v_o(u_b, h) = \frac{u_b}{l_r}(h_r - h)$	$\Xi(\nabla\phi, S) = \left\vert Q\frac{\partial \phi}{\partial s}\right\vert + \left\vert l_C \mathbf{q}_C \cdot \frac{\partial \phi}{\partial s}\right\vert$
Press-melt		$\Pi = -c_t c_w \rho_w \left(Q + l_C q_C\right) \frac{\partial p_w}{\partial s}$
Closure	$v_c(N, h) = \tilde{A} h \vert N\vert^{n-1} N$	$v_{cC}(N, S) = \tilde{A} S \vert N\vert^{n-1} N$



Sheet

$$\nabla . \boldsymbol{q} + v_o - v_c - m = 0$$

$$\int_{\Omega_i} \left[-\nabla\theta \cdot \boldsymbol{q} + \theta\,(v_o - v_c - m) \right] \,\mathrm{d}\Omega + \int_{\partial\Omega_i} \theta\, \boldsymbol{q} \cdot \boldsymbol{n} |_{\partial\Omega_i} \,\mathrm{d}\Gamma = 0$$

R-channels

$$\frac{\partial Q}{\partial s} + \frac{\Xi - \Pi}{\rho_i L} - v_{cC} - m_C = 0$$

$$\int_{\Gamma_j} \left[-\frac{\partial\theta}{\partial s} Q + \theta \left(\frac{\Xi - \Pi}{\rho_i L} - v_{cC} - m_C \right) \right] \mathrm{d}\Gamma + [\theta Q_j]_{-}^{+} = 0$$

add them and sum over all subdomains Ω_i and edges Γ_j

$$\sum_i \int_{\Omega_i} \left[-\nabla\theta \cdot \boldsymbol{q} + \theta\,(v_o - v_c - m) \right] \,\mathrm{d}\Omega + \sum_j \int_{\Gamma_j} \left[-\frac{\partial\theta}{\partial s} Q + \theta \left(\frac{\Xi - \Pi}{\rho_i L} - v_{cC} \right) \right] \mathrm{d}\Gamma + \int_{\partial\Omega_N} \theta\, q_N \,\mathrm{d}\Gamma = 0,$$

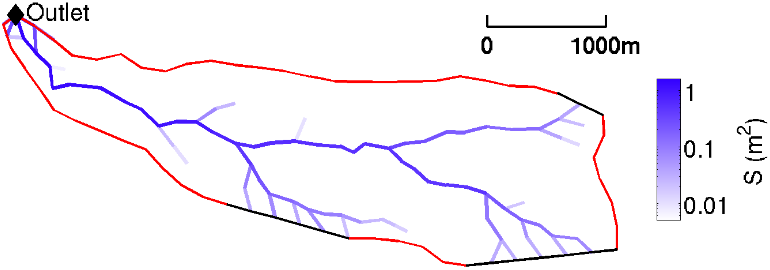
Mass exchange terms are magically taken care of! ODEs for h and S: $\frac{\partial h}{\partial t} = v_o - v_c$, $\frac{\partial S}{\partial t} = \frac{\Xi - \Pi}{\rho_i L} - v_{cC}$

Does the model converge

Question:

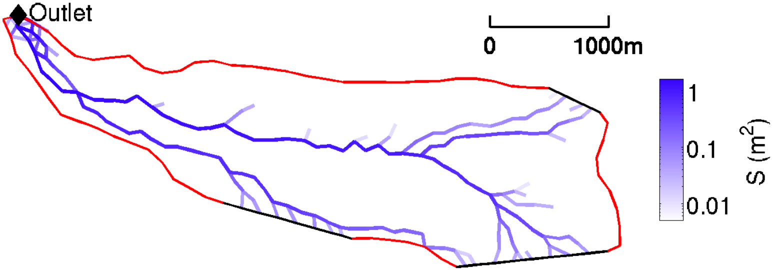
Does the predicted location of channels converge when the mesh is refined?
(Other types of convergence of course also interesting.)

Steady state channels



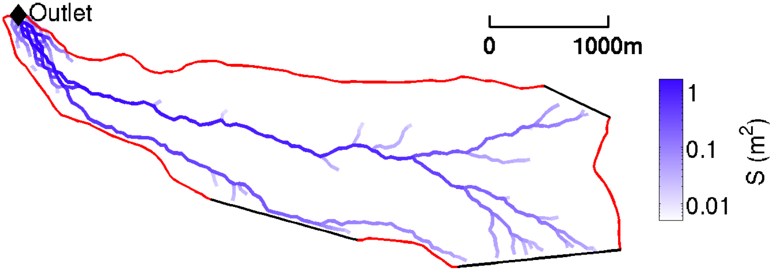
Mesh: 444 elements, 700 edges

Steady state channels



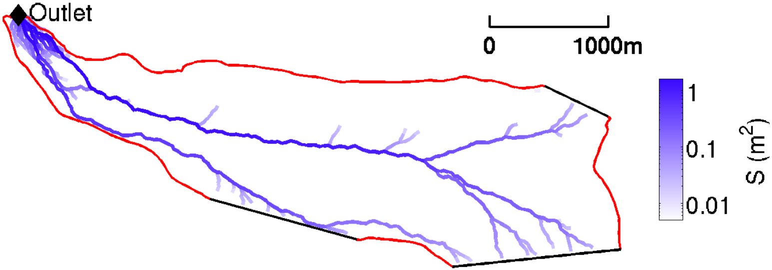
Mesh: 935 elements, 1458 edges

Steady state channels



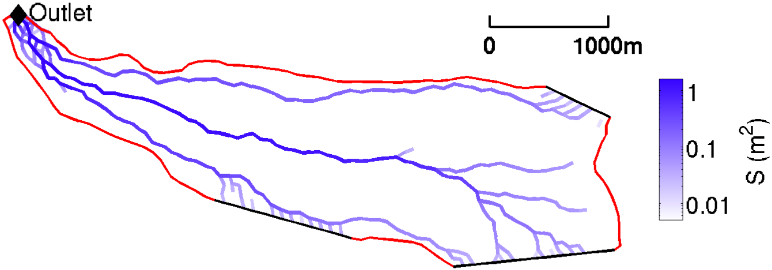
Mesh: 4502 elements, 6878 edges

Steady state channels



Mesh: 8933 elements, 13574 edges

Steady state channels



Mesh: 2203 elements, 3382 edges

Northern channel different!

Convergence under mesh refinement